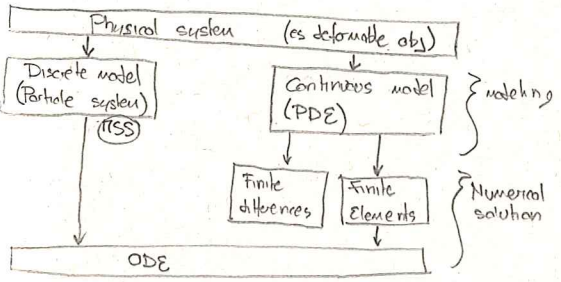
PBS SUMMARY

MASS-SPRING SYSTEMS (es cloth, hair, ...)

MSS



Deformable obj: • elastic • viscous • plastic } combination

① Spatial discretization

M	tot mass
n	# points
$m = M/n$	mass of each

- m_i mass
- $x_i(t)$ position
- $v_i(t)$ velocity

② Forces

external

- gravity: $F_i^g = m_i \begin{pmatrix} 0 \\ 0 \\ -9.81 \end{pmatrix} \frac{m}{s^2}$
- contact forces
- other (ron spring)

internal

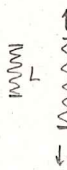
- elastic
- viscous damping

1D: $F = -k(l - L)$

3D: $F_i = -k(\|x_{ij}\| - L) \cdot \vec{x}_{ij}$

$\|x_{ij}\| = \|x_i - x_j\|$

$\vec{x}_{ij} = (x_i - x_j).\text{norm}$



energy

work = el. E $= \int_L^l k(x - L)\,dx$

$\mathcal{E} = W = \frac{1}{2} k (l - L)^2$

force $= -\frac{\partial \mathcal{E}}{\partial x_i}$

$(N \cdot m = J)$

→ result → $F_i = F_i^{int} + F_i^{ext}$ (sum all springs!)

dissipative (friction → thermal, $\mathcal{E}$ dissipates)

point damping: $F^{fd}(t) = -\gamma \cdot v(t)$ (γ is damp. coeff $\left[\frac{kg}{s}\right]$)

(damps all motion)

③ Dynamics

$x_i(t)$

$v_i(t) = \frac{d x_i(t)}{dt}$

$a_i(t) = \frac{d v_i(t)}{dt}$

$F_i = m_i \cdot a_i$

⇒ equations of motion

$$M \frac{d^2 x(t)}{dt^2} + D \frac{dx(t)}{dt} = F^{int}(t) + F^{ext}(t)$$

$3n$ 2nd order ODE

$M, D \in \mathbb{R}^{3n \times 3n}$ diagonal (m_i, γ)

④ Temporal Discretization

diff. eq → describes f using derivatives

ODE → contains only derivatives for 1 variable

es $x''(t) = \frac{F(t) - \gamma x'(t)}{m_i} \Rightarrow q''(x) = f(t, q, q')$ (↳ order of ODE)

IVP constants determined by $q(0) = \dots$

↳ ODE + initial val

MSS: we want $x_i(t)$

→ reduce to 1st order coupled ODE (we have initial values $q(0)$)

$q_i(t) = \begin{pmatrix} x_i(t) \\ v_i(t) \end{pmatrix}$ $\quad q_i'(t) = \begin{pmatrix} v_i(t) \\ \frac{F_i(t) - \gamma v_i(t)}{m_i} \end{pmatrix}$

Analytical sol: usually not available (would be perfect)

Numerical sol: compute approx for discrete timestep (time instants)

$q_n \rightarrow q_{n+1}$ always compute next

num. integration $q(t+h) = q(t) + \int_t^{t+h} q'(t)\,dt$

② NUMERICAL INTEGRATION

- accuracy → higher order explicit methods (how fast error decreases)
- stability → implicit method $|\gamma_n| < \sigma$

$h \to 0 \overset{?}{\Rightarrow}$ • convergence $q_i \to q(t_i)$ • efficiency

Approximations

$q(t)$	true sol
q_i	approx @ $t_i = t_0 + i \cdot h$
h	fixed timestep

Taylor exp. $q(t+h) = q(t) + \frac{q'(t)}{1!} \cdot h + \frac{q''(t)}{2!} h^2 + \dots$

$q(t+h) \cong q(t) + q'(t) \cdot h \Rightarrow q_{n+1} = q_n + h \cdot f(t_n, q_n)$

EULER'S METHOD

(order 1)

makes $\mathcal{O}(h^2)$ error

CODE

$x(t_0 + h) = x(t_0) + h \cdot v(t_0)$

$v(t_0 + h) = v(t_0) + h \cdot \frac{F(t_0) - \gamma \cdot v(t_0)}{M}$

$F(t)$ computed with $x(t)$ & ext. forces

problems: errors

- LOCAL $\mathcal{O}(h^{p+1})$ → order p
- GLOBAL (accumulates) → instability

HEUN'S METHOD

higher accuracy (truncate after)

... see how to compute mix derivates

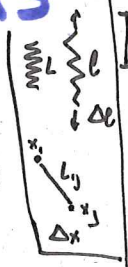
$q(t+h) \cong q(t) + \frac{h}{2}(q'(t) + q'(t+h))$ use

PBS

1. MASS SPRING SYSTEMS

m, $x(t)$, $v(t)$

Hook's law 1D: $F = -k\Delta\ell$ $\quad (\ell - L) = \Delta\ell$



2D: $f_{ij} = -k\left(\|x_i - x_j\| - L_{ij}\right)\frac{\vec{x}_i - \vec{x}_j}{\|x_i - x_j\|}$

$F_i = \sum_j f_{ij}$ (i gets from j)

$W_{ij} = \int_{L_{ij}}^{\ell} k(x - L_{ij})\,dx \qquad \mathcal{E} = W = \frac{1}{2}k\Delta x^2$

$f_i = -\frac{\partial E}{\partial x_i}$ $\qquad F_i^{diss} = -\gamma v_i(t)$ point dampers

$f_i = m_i \ddot{x}_i \Rightarrow f_{spr} = \Pi\ddot{X} + D\dot{X}$ $\qquad \Pi = $ $3N +$ D viscous coeff

$\Rightarrow$ $\quad S = \begin{pmatrix} X \\ V \end{pmatrix} \quad S' = \begin{pmatrix} V \\ \Pi^{-1}(F - DV) \end{pmatrix}$

$X(t+\Delta t)$

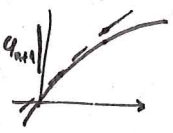
$V(t+\Delta t)$

NUM. mt of ODE

h timestep

Euler method $\quad q(t+h) = q(t) + h\cdot\frac{dq(t)}{dt}$ $\quad$ order 1

- Accuracy (error) $\quad \Delta_{lc} = \left|\left(q_n + \int_{t_n}^{t_{n+1}} q'(t)\,dt\right) - q_{n+1}\right|$



$\Delta_{glb} = |q_i - q(t_i)|$

error depends on step size h: $O(h^{p+1})$ → order p ($h_1 > h$ = good)

$\Delta_{glo}^{est} = N\cdot\Delta_{lc} \qquad N = \frac{T}{h}$ or $O(h^p)$

- Stability (not → ∞) $\quad$ ex $q' = \lambda q \quad \lambda < 0 \Rightarrow q(t_n) = (1+\lambda\Delta t)^n q_0$

$\Rightarrow \Delta t < \frac{2}{|\lambda|}$ ($|1+\lambda\Delta t| < 1$)

- Convergence $\quad h \to 0 \overset{?}{\Rightarrow} q_i \to q(t_i)$
- Efficiency $\quad$ good choice for problem?

Taylor exp $\quad q(t+h) = q(t) + \frac{q'(t)}{1!}h + \frac{q''(t)}{2!}h^2 + \dots O(h^?)$

Heun's method order 2

$q(t+h) \approx q(t) + \frac{h}{2}\left(q'(t) + q'(t+h)\right)$

$\tilde{q}_{n+1} = q_n + h f(t_n, q_n)$ // euler

$q_{n+1} = q_n + \frac{h}{2}\left(f(t_n, q_n) + f(t_{n+1}, \tilde{q}_{n+1})\right)$ $\quad$ unknown: use Euler step

Explicit midpoint order 2 $\quad$ use $f(t + \frac{h}{2})$ $\quad q(t+h) = q(t) + hq'(t+\frac{h}{2})$

$\tilde{q}_{n+1/2} = q_n + \frac{h}{2}\cdot f(t_n, q_n)$ // euler

$q_{n+1} = q_n + h\cdot f(t_{n+\frac{1}{2}}, \tilde{q}_{n+\frac{1}{2}})$ $\quad$ 2nd order due to 2 eval of funct $(+O(h^3))$

RK4 most popular

4 step eval → 4th order accuracy

$q_{n+1} = q_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$

Newton's method △ not integration

$q_{n+1} = q_n + hf(t_{n+1}, q_{n+1}) \Rightarrow 0 = q_{n+1} - \dots = g(q_{n+1})$

guess & correct

$g'(\tilde{q}_{n+1})\cdot\Delta q = -g(\tilde{q}_{n+1})$

Implicit Euler (backward) slope at end $\quad q(t+h) = q(t) + h\cdot\underbrace{f(t_{n+1}, q_{n+1})}_{q'(t+h)}$

stability $\quad (1 - \lambda\Delta t)^{-1} < 1 \Rightarrow$ stable for all

semi-impl

Simplectic Euler 2nd order (error $O(h^4)$) explicit euler, swap order of comp $V(t+h)$, $X(t+h)$ (use new velocity)

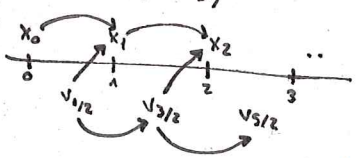
energy cons

Verlet $\quad X(t+h) + X(t-h) = \dots$ (taylor)

$X(t+h) = 2X(t) - X(t-h) + h^2\ddot{X}(t) + O(h^4)$

Leapfrog integration 2nd order $\quad V(t+\frac{h}{2}) = V(t-\frac{h}{2}) + h\cdot a(t)$ $\quad a = x''$

$x(t+h) = x(t) + hV(t+\frac{h}{2})$



constraints

$C(x_1 \dots x_n) = 0$ (satisfied)

x_1 pos
x_1, x_2 length
x_1, x_2, x_3 area
x_1, x_2, x_3, x_4 vol

soft

→ pot $\mathcal{E}$

$\mathcal{E}_c(x_1 \dots x_n) = \frac{1}{2}k\,C(x_1 \dots x_n)^2$

$F_{penalty} = -\nabla\mathcal{E}_c \Rightarrow f_j = -\frac{\partial\mathcal{E}_c}{\partial x_j} = -kC\,\partial$

hard

integration schemes recap

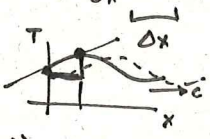
2. PDE

Advection eq. $u_t + c u_x = 0$ $\quad \Delta x = h \neq \Delta t$

$T_0(x)$ c $\Rightarrow T(x,t)$

$T(x, t+h) = T(x - c\cdot h, t) = T(x,t) - \frac{\partial T}{\partial x} c\cdot h + O(\Delta x^2 \& h^2)$

$\frac{\Delta T}{\Delta t} \xrightarrow[\Delta t \to 0]{} \frac{\partial T}{\partial t} = -c \cdot \frac{\partial T}{\partial x}$ PDE solve...



$f(x - ct) \rightarrow$ initial condition $\Rightarrow T_0(x - ct)$

IVP: need initial condition $T(x,0) = T_0(x)$ $\quad u_t = c^2 \Delta u$

Classification $\quad A u_{xx} + 2B u_{xy} + C u_{yy} = f(x, y, u, u_x, u_y)$

$B^2 - AC > 0$ hyperbolic → wave eq $\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \nabla^2 u$

$= 0$ parabolic → heat $\frac{\partial}{\partial t} u(x,t) = a \nabla^2 u$

< 0 elliptic → poisson eq $-\Delta u(x,y) = f(x,y)$ general

∇ (nabla) $= \left(\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n}\right)^T$

Δ (Lapl) $= \nabla_x^2$

order = order of highest partial derivative

linearity =

FD

use uniform grid

$\left.\frac{\partial u(x,t)}{\partial t}\right|_{\substack{x=x_0 \\ t=t_0}} \approx \frac{u(x_0, t_0 + \Delta t) - u(x_0, t_0)}{\Delta t}$

discretize in finite points

boundary conditions

1D

$-u_{xx} = f \quad x \in \Omega$
$u(0) = u(1) = 0$

$u_{xx}(i) \approx \frac{u_x(i) - u_x(i-1)}{h} = \frac{u(i+1) - 2u(i) + u(i-1)}{h^2}$

$Au = f$

$A = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & \ddots & \\ & & & \end{bmatrix}$

solve with conjugate gradients

FEM

assume PDE satisfied pointwise

$-u_{xx} = f$

$\int_\Omega \cdot v \rightarrow$ P.I. → weak form (1) $\int_0^1 u_x v_x = \int_0^1 f v$ (it's 1st order)

$\int f' g = fg - \int f g'$

Ritz galerkin (2) (discretization)

$u(x) \approx \sum_i u_i N_i(x)$ $\quad \mathbb{R}^1 \to \mathbb{R}$

nodal coeff, Basis fct

$\rightarrow \frac{\partial u}{\partial x} = \sum_i u_i \frac{\partial}{\partial x} N_i$

linear interp

insert into weak

$\int (\sum_i u_i N_i')(v') = \int f v$

$\sum_j v_j \int (\sum_i u_i) \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dx = \sum_j v_j \int f N_j dx$

$\Rightarrow Ku = f$

spd, sparse

$f_i = \int_\Omega f N_i(x) dx$

$K_{ij} = \int_\Omega \frac{\partial}{\partial x} N_i \frac{\partial}{\partial x} N_j dx$

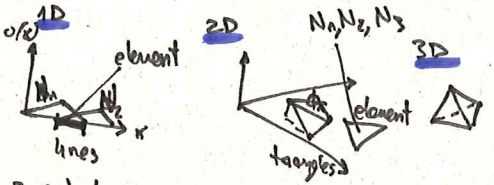
Steps FEM

1. weak formulation
2. R-G approach
3. choice of fh space
4. geom decomp of domain
5. setup matrix eq
6. solve lin. syst

Linear finite elements (3) types of FE

- smooth (once differentiable)
- simple
- $N_i(x_j) = \delta_{ij}$
- compact support // piecewise

→ piecewise linear basis fct



2-node line seg with 2 lbf (linear basis fct)

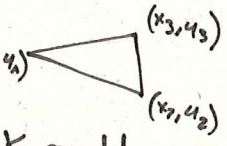
3-node triangle with 3 lbf

4-node tetrahedron with 4 lbf

$N_i(x,y) = a_i x + b_i y + c_i$ // plane

2D: for each triangle basis

$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \end{bmatrix}$



→ K computed

Tessellation (4) 2D delaunay triangulation / 3D tetrahedral mesher

Evaluate integrals (5)

RHS $\quad f_i = \int_\Omega f N_i \, dx dy = \sum_{e=1}^{n_{ei}} \int_{\Omega^e} f N_i^e$

zero everywhere else

$\approx f(x_q, y_q) \cdot N_i^e(x_q, y_q) \cdot A_e$

A_e (area) $\quad \Omega_i$ (all) $\quad \Omega^e$ (single) $\quad x_q$ barycenter

LHS...

(6) solve $Ku = f$

$f(\vec{x}_q) N_i^e(\vec{x}_q) A_e$ (an average)

element centered eval

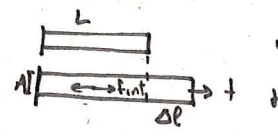
$K_{ij} = \sum_{e \in S_{ij}} \int_{\Omega^e} \frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} dx dy$

$u_{tt} = c^2 \nabla^2 u$

$u_{tt} = c^2 \Delta u$

5. Continuum Mech. & FEM

min E_{pot}

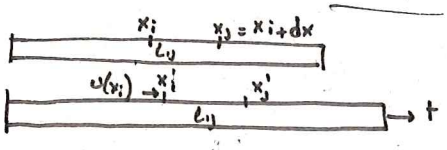


1D

stretch strain $\varepsilon = \frac{\Delta l}{L} = \frac{\partial u}{\partial x}$ ← material const

force density stress $\sigma = \frac{f_{int}}{A}$] Hooke $\sigma = k\varepsilon$ $(F = -k\Delta l)$

Ⓔ density $\psi = \frac{1}{2} k \varepsilon^2 = \sigma\varepsilon$



$u(x) = x' - x$

$\Delta l = l - L$

lim $\varepsilon(x^*) = \frac{\partial u}{\partial x}\Big|_{x^*}$ $dx \to 0$

strain Ⓔ $W_i = \psi_i \cdot L_i$

tot $W = \sum_i W_i$

discretize into FE $\varepsilon_i \approx \frac{(x_{i+1}' - x_i') - L_i}{L_i}$

min part Ⓔ $\frac{\partial W_i}{\partial x_i} = -k\varepsilon_i \doteq 0$ $f_{penalty\,i} = -\frac{\partial W_i}{\partial x_i}$

$\Rightarrow n-1$ eq/unknowns

Continuous Elasticity 3D

undef $\Omega \subset \mathbb{R}^3$ $\vec{x}$

def Ω' $\vec{x}'$

$u: \Omega \to \Omega'$

$\vec{x}' = \vec{x} + u(\vec{x})$ $\to \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

deformation gradient

$d = \vec{x}_2 - \vec{x}_1$

$d' = x_2' - x_1' = \dots$

$= d + u(x_1 + d) - u(x_1)$

Taylor exp $\approx u(x_1) + d \cdot \nabla u(x_1)$

$\Rightarrow \boxed{d' = (I + \nabla u)d = F d}$ def gradient

$\nabla u = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}_{3\times 3}$

Non-lin strain

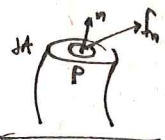
(def = change in length in all dir)

$|d'|^2 - |d|^2 = d^T (F^T F - I) d$

Green strain (not invar) $\mathcal{E} = \frac{1}{2}(F^T F - I) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T \nabla u)$ Cauchy

$\Rightarrow \mathcal{E} = \frac{1}{2}(F^T F - I)$

3D stress



traction force $t_n(\vec{n}, \vec{p}) = \frac{f_n}{dA}$ $(dA \to 0)$

$= \sigma(x) \cdot n$

$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau & \sigma_y & \tau \\ \tau & \tau & \sigma_z \end{bmatrix}$ Cauchy stress tensor

Hooke model

Ⓔ density $\psi = \frac{1}{2}\lambda\,\mathrm{tr}(\varepsilon)^2 + \mu\,\mathrm{tr}(\varepsilon^2)$ (perturb J change, perturb strain)

Cauchy stress $\sigma = \frac{\partial \psi}{\partial \varepsilon} = \lambda\,\mathrm{tr}(\varepsilon) I + 2\mu\varepsilon$ | λ, μ material consts

FE discretization

$N_i(\vec{x}) = a_i x + b_i y + c_i z + d \to$ x_A

$f = \frac{\partial f}{\partial x} \overset{FEM}{=} \sum_j f \frac{\partial N_j}{\partial x}$ $\quad \omega^e = \int_{\Omega^e} \psi(F) = \psi(F)\bar{V}^e$

Linear elasticity method:

find Δx st $f(x + \Delta x) = f(x) + K\Delta x + O(\Delta x^2) = 0$

$f(\vec{x}^*) \neq 0$

K stiff matrix $K = \frac{\partial f}{\partial x} \in \mathbb{R}^{12\times 12}$ | solve $K\Delta x = f$

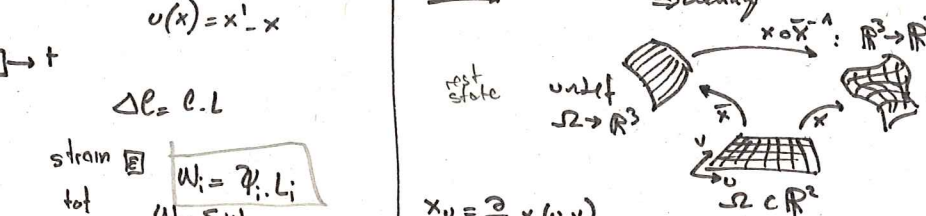
6. Thin shells (& rods)

- stretching ↔ tension, compr
- bending ↷ ⇒ buckling (in-plane def) (out of pl def)

mid S

mids = plane of 0 stress

only dep on thickness



$x_u = \frac{\partial}{\partial u} x(u,v)$

$\bar{x}_v = \dots$

$\vec{n} = \frac{x_u \times x_v}{\|x_u \times x_v\|}$

Kirchoff Love KL ass:

kinem. assumptions

Stretch

$d = u_2 - u_1$ infinitesimal

$d' = x(u_2) - x(u_1) \approx J d$

$ds^2 = d'^T d' = d^T J^T J d$

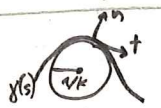
$x(u_1 + d) \approx x(u_1) + d \cdot \nabla x(u_1)$ T. Exp

$J = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \\ \partial z/\partial u & \partial z/\partial v \end{pmatrix}_{3\times 2}$

1st FF $I = \begin{pmatrix} x_u x_u & x_u x_v \\ x_v x_u & x_v x_v \end{pmatrix}$

$ds' - ds = \dots = d^T (I - \bar{I}) d$ difference of 1st FF captures in-plane def

Bending



$t = \gamma'$ $(\|1\|)$

$n = t'$ "

$k = |t'| = \gamma'' \cdot n$

$k = \frac{dt}{ds}$ how much tangents change in normal dir when changing point on curve

2nd FF $II = \begin{pmatrix} x_{uu} \cdot n & x_{uv} \cdot n \\ x_{vu} \cdot n & x_{vv} \cdot n \end{pmatrix}$

curvatures

- normal $\frac{t^T II\, t}{t^T I\, t}$
- mean $H = \frac{1}{2}(k_1 + k_2) = \mathrm{tr}(I^{-1} II)$
- principal $k_1 = \max_t k(t)$, $k_2 = \min_t$
- gauss $K = k_1 \cdot k_2 = \det(I^{-1} II)$

$\mathcal{E}_S = \int_\Omega k_s \|I - \bar{I}\|^2 + k_b \|II - \bar{II}\|\, J_\Omega$ (in-plane def, bending def, determine def up to rigid motion)

discretize

$\mathcal{E}_S$ (don't discretize continuous but define discrete (FD/FEM))

$x(u,v) = \sum_i x_i N_i$ (FE) ↯

↳ quadratic | extra DoF

↳ triangles (have differentiable)

in-plane $\mathcal{E}_L = k_{stretch} \cdot \sum_{e_{ij}} |\bar{e}_{ij}|^2 \left(\frac{|e_{ij}|}{|\bar{e}_{ij}|} - 1\right)^2$ length change

$\mathcal{E}_A = k_{area} \cdot \sum_{f_{ijk}} |\bar{f}_{ijk}| \left(\frac{f_{ijk}}{\bar{f}_{ijk}} - 1\right)^2$ area

bending $\mathcal{E}_B = k_{bend} \cdot \sum_{e_{ij}} 2(\theta_{ij} - \bar{\theta}_{ij})^2 \frac{\|\bar{e}_{ij}\|}{\bar{h}_e}$

invariant for in-plane def

$\mathcal{E}_{DS} = \mathcal{E}_L + \mathcal{E}_A + \mathcal{E}_B$

rods skipped

PBS: rigid body sim

↱+CG 6.2

$p(t) = R(t)\,p_0 + x(t)$ — 1 representation

$x \in \mathbb{R}^3$, $R \in \mathbb{R}^{3\times3}$

$r_i(t)$ = position of particle (world coord) ? global

$CoM = \frac{\sum m_i r_i}{m}$

(linear velocity) $v = x'$ — R fixed — 2 Kinematics → $R' = \omega * R$ (cot change) orientation

(angular velocity) col.wise × (·x) — $R = [x'\ y'\ z']$

(pos change)

axis & speed ($\omega \in \mathbb{R}^3$)

$r(t) = o + b$

$r'(t) = \omega(t) \times b(t) = \omega \times r$

dynamics

$F = \sum F_i$ (force)

$T = \sum T_i$ (torque)

$T_i = (r_i - x) \times F_i$

$q' = \frac{1}{2}(1^{-1}L)q$ (ω)

$F = ma = mv' = p'$

$P = mv$ (lin mom) [Ns]

$L = I\omega$ (angular mom)

↳ inertia tensor $\in \mathbb{R}^{3\times3}$ (mass distribution) relative to CoM

$P' = F$

$L' = T = I\omega'$

$F \to P \to v \to x$

orthogonality: P indep R, ω; L indep x, v

→ * how to (pre)-compute it

inertia: momentum is conserved

$I_{body} = \sum_i m_i((r_i^T r_i)1 - r_i r_i^T)$

$I(t) = R \cdot I_{body} \cdot R^T$

state $S(t) = \begin{pmatrix} x \\ R \\ P \\ L \end{pmatrix}$ $S'(t) = \begin{pmatrix} v \\ \omega \times R \\ F \\ T \end{pmatrix}$

ODE: $x' = \ldots$, $R' = \ldots$

Simulation algo

precompute x, M^{-1}, I_{body}^{-1}

kin: how it moves

dyn: what causes it

compute: $F \leftarrow \sum F_i$, $T \leftarrow \sum (r_i - x) F_i$ — $F = \sum$, $T = \sum$

update (euler): $x \leftarrow x + \Delta t\, M^{-1} P$; $P \leftarrow P + \Delta t F$; $R \leftarrow R + \Delta t (\omega * R)$; $L \leftarrow L + \Delta t T$; $I^{-1} \leftarrow R\, I_{body}^{-1} R^T$; $\omega \leftarrow I^{-1} L$

→ $x \mathrel{+}= \Delta t\, v = \Delta t (M^{-1} P)$; $R \mathrel{+}= \Delta t R' = \Delta t (\omega * R)$

$P \mathrel{+}= \Delta t F$; $L \mathrel{+}= \Delta t T$

$I^{-1} = R\, I_{body}^{-1} R^T$ (rot); $\omega = I^{-1} L$

$N = \frac{M}{kg}$, S^2

per-particle quantities: $r_i \leftarrow x + R r_{i0}$; $v_i \leftarrow v + \omega \times (r_i - x)$ — $r_i = x + R\, r_0$; $v_i = v + \omega \times (r_i - x)$

quaternions remember (t) (t)

Collision Handling

$J = \int F\, dt$ * complete

impulse $J = m \Delta v$; $\Delta P = J$; $J = j\hat{n}$ (j = scalar)

$\Delta L = T_{impulse} = (p - x(t)) \times J$] any mom

$v_{rel,aft} = -\varepsilon v_{rel,bef}$

stops ↙ ↘ bouncy

$\varepsilon \in [0,1]$ (restitution coefficient)

resting contact ∀contact p ⇒ $v_{rel} = 0$

$v_{rel} = \hat{n}(v_a - v_b)$ (normal of contact pt) > 0, $= 0$, < 0

impulse driven (manip. ij)

force driven (manip. Q)

Integration methods : symplectic

Notation

$\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$ $\nabla v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

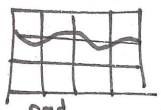
$\nabla \cdot v = v_x + v_y + v_z$

$\Delta = \nabla^2 v = \nabla \cdot \nabla v = v_{xx} + v_{yy} + v_{zz}$

Fluid simulation

2D, 3D

Eulerian (viewpoint) — Lagrangian — hybrid FLIP



grid — particles

wanted: u, v, w

material derivative: $\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu \Delta \vec{v} + \vec{g}$ — pressure ② — viscosity ② — ② // momentum conservation — ΣF_i

∇ node

Δ laplace $= \nabla \cdot \nabla$ (dweg)

$\nabla \cdot \vec{v} = 0$ // mass conserv (div free)

v = flow velocity; ρ = density = M/V; ν = viscosity = $\frac{\mu}{\rho}$

what is $\frac{Dv}{Dt}$?

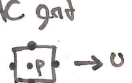
mat. der $\frac{dq(x,t)}{dt} = \frac{\partial q}{\partial t} + \frac{\partial q}{\partial x}\frac{dx}{dt} = \frac{\partial q}{\partial t} + \nabla q \cdot v$

① advection
② body force (external)
③ diffusion / viscosity
④ pressure solve
⑤ apply pressure

→ $\begin{cases} v_{n+1} = v_n' - \Delta t \frac{1}{\rho}\nabla p \\ \nabla \cdot v_{n+1} = 0 \end{cases}$ (solve for p) — intermediate

semi Lagrangian operator splitting

MAC grid



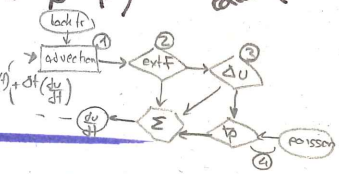
2nd order acc

poisson eq

$v_{n+1} = v_n + \Delta t\left(\vec{g} + \nu \nabla^2 \vec{v}_n - \frac{1}{\rho}\nabla p\right)$ update rule — with correct p

$Ap = d$

how to compute



Lagrangian

no advection
no incompressibility

$\rho \frac{\partial \vec{v}}{\partial t} = -\Delta p + \mu \Delta v + \rho g$ — f_p f_μ f_{ext} — $\rho a = \frac{F}{V} = f$ // force densities

SPH (smoothed part hydrodynamics)

continuum assumption
smoothing kernel W

properties: $\int W(x - x', h)\,dx' = 1$

$\lim_{h \to 0} W(x - x', h) = \delta(x - x')$ // dirac

$\|x - x'\| > h \to W(x - x', h) = 0$ // local

$W(x_j, H)$ — // no mol

$\nabla A(x) = \sum_{j \in N} \frac{m_j}{\rho_j} A_j \nabla W(x - x_j, h)$ — V_j

N includes self — // $\nabla \to \nabla$ kernel

ideal gas state eq $p = k\rho$ → $P_i = k(\rho_i - \rho_0)$ — stiffness — rest density (water = 1000)

→ clamp [0, ∞[stability, no attraction

$f_p = -\nabla p$ // pressure force

$k \nearrow \to \Delta t \searrow$

comp order
1. n search
2. ρ, p
3. body forces
4. diff / v
5. press force / v
6. update v, x
7. collision

Symmetry

Newton's 3rd law: actio = reactio

sol: pressure average $\vec{f}_{p,i} = -\sum_j \frac{m_i}{\rho_i}\left(\frac{p_i + p_j}{2}\right)\nabla W(i,\cdot)$

force — $F = \nabla(\vec{f})$ force density

$\vec{f}_{\mu,i} = \mu \sum_j \frac{m_j}{\rho_j}(v_j - v_i)\,\Delta W(\ldots)$

↳ instead of (v_j): visc. force ∝ velocity difference

$f_{g,i} = \rho_i g$

$f_{tot,i} = f_{p,i} + f_{\mu,i} + f_{ext,i}$ — $a_i = \frac{f_i}{\rho_i}$ → ∝ ? to V

explicit integr: $v_i(t + \Delta t) = v_i(t) + \Delta t \cdot a_i(t)$

(semi-implicit) $x_i(\ ) = x_i + \ldots \cdot v_i(t + \Delta t)$ (symplectic)

Neighbor search

• grid cell tech — h×h×h → 27 cells — $O(\frac{N^2}{\pi})$ ≙ $O(N)$ — #cells

$\pi \propto L^3 \propto N$

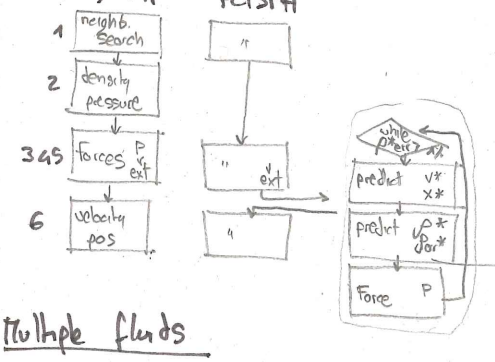
WC SPH (weakly compressible) k large (stiffness) ⊖ cost

ISPH (incompressible) poisson eq, large timestep ⊖ cost

PCISPH (prediction-correction - incompressible) ⊕ low-cost

iterative until divergence free

(WC)SPH PCISPH



$f_v = f_n$

★ SOLIDS MELTING

Multiple fluids

SPH must adapt interface discontinuities

≠ densities

$\tilde{\rho}_i = m_i \delta_i = m_i \underbrace{\sum_j W(..)}_{\text{particle density}}$ // assume all ◻ have same mass

$\tilde{F}_i = -\frac{\nabla \tilde{p}}{\delta}$ adapted forces

diffusion eq

$\frac{\partial A}{\partial t} = c \nabla^2 A$

Turbulence

tries to reinject lost ◻ (due to imprecise advection/diffusion) → semi Lagrange / MacCormack

resolution tradeoff

quality ⟷ efficiency

vorticity: $\omega = \nabla \times \vec{v}$ $\vec{n} = \frac{\nabla \eta}{|\nabla \eta|}$, $\eta = |\omega|$

$f = c\, \vec{n} \times \omega$ // force in tangent dir

c ↓ strength

2D:

wave eq: only vertical velocity field

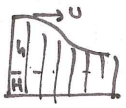
Sim. speed up
- GPU, multicore, DS
- large timestep
- multi-scale methods
- coarse sim + details

SWE (Shallow water eq)

top view

$\frac{Dh}{Dt} = -h \nabla \cdot \vec{v}$

$\frac{D\vec{v}}{Dt} = -g \nabla (h + H) + \vec{a}_{ext}$



$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \nabla q \cdot \vec{v}$

FD

bound: mirror v

advect h
advect v_1, v_2
update h
update v

3x interp backtrack interp write

tradeoff

Boundary force (artifacts)

no particles → incorrect density/ρ

↑ add n particles (stable)

Collision Detection

- compute dyn
- detect coll
- resolve coll

Broad phase Culling → model partitioning (OBJ)

BV – sweep & prune (AABB) proj/sort/prune $O(n \log n + k)$

- sphere
- AABB
- OBB
- k-DOP
- c. hull

fast ↓ precise

// tight ~ accurate

Δ close proximity

Narrow ph:
- cull non-colliding regions
- primitive tol test (Δ)

sphere $|c_1 - c_2| > r_1 + r_2$ // no collision

AABB c_i, r_x, r_y " V (or) 2D: 4-DOP 3D: 6-DOP

k-DOP · halfspaces, $\vec{n}$ in k orientation → $\frac{k}{2}$ proj

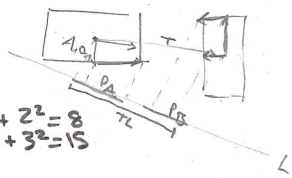
OBB $v \in \mathbb{R}^3$ $\mu = \frac{1}{n} \sum v_i$ mean

$C_{jk} = \frac{1}{n} \sum_i \bar{v}_j \bar{v}_k$ ($\bar{v}_i = v_i - \mu$)

cov. matrix $\in \mathbb{R}^{3\times3}$ → EV = axes

$P_A = |a_1 A_1 L| + |a_2 A_2 L|$ (radii, axis)

$\exists L: |T \cdot L| > P_A + P_B$



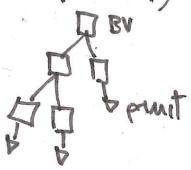
SAT (sep. ax test)

2D: $2+2+2^2 = 8$

3D: $3+3+3^2 = 15$

VR (voronoi reg)

BVH (hierarchy) tree: top-down bot-up



Deformation → BDT (bounded def tree)

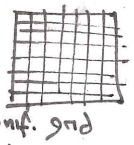
bottom-up update

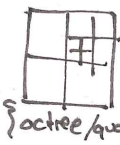
layered vs wrapped

Spatial partitioning (SPACE) ⊕/⊖

BVH fits obj red. space
- update deform

SP fits space redund. obj
+ rigid/deform
+ efficient





} obj dependent

cell size

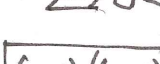
#cells ↑ #prim/cell ↓ (increase)

tradeoff

Hash (cell) → index

1) v
2) tetrahedra (bb)
3) (no) collision/self

Particle collision



project → check if inside triangle

$d = N \cdot (x - p_0)$
if $(d < 0)$ $x -= N * d$
proj

$(x - v_0) \cdot ((v_2 - v_0) \times n) > 0$

do for 3 v

$d = (x - P)^2$
if $(d < r^2)$ // collision

⊕